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# On the Number of Characters in a $p$-Block of a $p$-Solvable Group

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Let $G$ be a finite group and let $p$ be a prime divisor of $|G|$. Let $B$ be a $p$-block of irreducible characters of $G$ and let $d$ be the defect of $B$. Brauer conjectured in [1] that $B$ contains at most $p^d$ complex irreducible characters and in a joint work with Feit [2], he showed that $B$ certainly contains no more than $\frac{1}{4}p^{2d} + 1$ such characters. The proof of this result is purely arithmetic and makes no use, for example, of properties of defect groups. Brauer's conjecture is known to be true for defect groups whose structure is not too complicated, abelian $p$-groups with at most two generators and dihedral 2-groups being examples.

There are reasons to hope that the Brauer–Feit upper bound can be improved for $p$-solvable groups, as Fong has shown in [7] and [8] how many problems in the modular representation theory of these groups can be reduced to the analysis of certain minimal configurations. Indeed, using Fong's methods, Nagao proved in [12] that Brauer's conjecture holds for all $p$-solvable groups provided that it holds for a group $G$ of the form $G = UH$, where $U$ is a normal elementary abelian $p$-group and $H$ is a $p'$-group which acts faithfully and irreducibly on $U$.

In this paper, we extend Nagao's work to show that there is an integer-valued function $f$ with the property that if $B$ is a $p$-block of defect $d$ of any $p$-solvable group, then $B$ contains at most $f(d)\,p^d$ irreducible characters. The function $f$ that appears here can be taken to be the best of the various functions arising in the statement of Jordan's theorem on the index of normal abelian subgroups in finite complex linear groups. While this result is only of qualitative interest, it is at least in keeping with Brauer's conjecture, especially as $p$ becomes large. By restricting attention to smaller classes of $p$-solvable groups, the function $f$ can be much improved. We show that Brauer's conjecture holds for a group of odd order whose Sylow $p$-complement is nilpotent.

## 1

We begin by recollecting results of Gallagher [9, 10] on the number of conjugacy classes in a finite group. Four pertinent facts taken from his papers form our first lemma.



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LEMMA 1.1 (Gallagher). *Let $G$ be a finite group and let $k(G)$ denote the number of conjugacy classes of $G$.*

(a) *If $N$ is a subgroup of $G$, $k(G) \leqslant k(N) \mid G : N \mid$.*

(b) *If $N$ is a normal subgroup of $G$, $k(G) \leqslant k(N)\, k(G/N)$.*

(c) *If $N$ is a normal subgroup of $G$ and $\lambda$ is an irreducible $G$-invariant character of $N$, the number of irreducible constituents of $\lambda^G$ does not exceed $k(G/N)$.*

(d) *If $g$ is an element of $G$ that satisfies $\chi(g) \neq 0$ for all irreducible characters $\chi$ of $G$, $k(G) \leqslant \mid C_G(g) \mid$.*

We now derive some consequences of these results. Let $S$ be a class of $p$-solvable groups that is subgroup and factor group closed. We will call a group $G$ in $S$ *primitive* if it has the form $G = UH$, where $U$ is a normal elementary abelian $p$-group and $H$ is a $p'$-group that acts faithfully on $U$. Using this notation, we have the following lemma.

LEMMA 1.2. *Suppose that any primitive group $M$ in $S$ satisfies the condition*

$$k(M) \leqslant f(b)\, p^b, \tag{*}$$

*where $f$ is some increasing function defined on the positive integers, and $p^b = \mid M \mid_p$. Then any group $G$ in $S$ in which $O_{p'}(G) = 1$ also satisfies (*).*

*Proof.* Let $p^a = \mid G \mid_p$. Let $N = O_p(G)$ and let $H$ be a Sylow $p$-complement. Let $V$ be the Frattini subgroup of $N$. Since $O_{p'}(G) = 1$, it follows from the Hall–Higman lemma [11, 6.5, p. 690] that $N/V$ is a normal elementary abelian self-centralizing subgroup of $NH/V$. Thus $NH/V$ is a primitive group in $S$ and consequently we have

$$k(NH/V) \leqslant f(c)\, p^c,$$

where $p^c = \mid N : V \mid_p$. It follows from 1.1(b) that

$$\begin{aligned} k(NH) &\leqslant f(c)\, p^c k(V) \\ &\leqslant f(c) \mid N \mid. \end{aligned}$$

Lemma 1.1(a) now implies that

$$\begin{aligned} k(G) &\leqslant k(NH) \mid G : NH \mid \\ &\leqslant f(c)\, p^a \leqslant f(a)\, p^a, \end{aligned}$$

and the result is thus proved.

Continuing with the same notation, we show next how the above ideas can be combined with the methods of Fong to obtain information about the number of irreducible characters in a $p$-block of a group in $S$.

THEOREM 1.3. *Suppose that the hypothesis of Lemma 1.2 holds. Let $B$ be a $p$-block of defect $d$ of a group $G$ in $S$ and let $k(B)$ be the number of irreducible characters in $B$. Then $k(B) \leqslant f(d)\,p^d$.*

*Proof.* We proceed by induction on $|G|$. Let $p^a = |G|_p$. Following Fong's reduction technique, outlined in [7, p. 278], we can assume that $d = a$ and that $Z = O_{p'}(G)$ is central in $G$. Moreover, there exists an irreducible character $\lambda$ of $Z$ such that the irreducible characters of $B$ are precisely the irreducible constituents of $\lambda^G$. It follows from 1.1(c) that $k(B) \leqslant k(G/Z)$. Since $O_{p'}(G/Z) = 1$, 1.2 shows that $k(G/Z) \leqslant f(a)\,p^a$ and our theorem is thus proved.

Our next objective is to find a function $f$ in 1.2 that will apply to all $p$-solvable groups. We recall from Jordan's theorem that there is an increasing function $f$ defined on the positive integers such that if $H$ is a finite complex linear group of degree $n$, $H$ contains a normal abelian subgroup of index not exceeding $f(n)$. (An explicit form of $f$ is given in [5, 30.4, p. 177].)

LEMMA 1.4. *Let $G = UH$, where $U$ is a normal self-centralizing elementary abelian subgroup of order $p^n$ and $H$ is a $p'$-subgroup. Let $f$ be a function of the type occurring in the statement of Jordan's theorem. Then $k(G) \leqslant f(n)\,p^n$.*

*Proof.* $H$ is faithfully represented on $U$ as a linear group of degree $n$ over $GF(p)$. Thus since $H$ has order coprime to $p$, $H$ has a faithful complex representation of degree $n$. It follows that there is a normal abelian subgroup $A$ of $H$ with $|H : A| \leqslant f(n)$. (We will not actually use the fact that $A$ is normal.)

Using the fact that $U$ is completely reducible as an $A$-module, together with Schur's lemma, it is easy to show by induction on $n$ that there is an element $u$ of $U$ that is centralized by no nonidentity element of $A$. Thus $|C_G(u)| \leqslant f(n)\,p^n$.

As $G$ has a normal abelian Sylow $p$-subgroup, it follows from Ito's theorem [6, 9.13, p. 56] that the degrees of the irreducible characters of $G$ are coprime to $p$. We deduce from [6, 6.4, p. 34] that $\chi(u) \neq 0$ for any irreducible character $\chi$ of $G$. We now apply 1.1(d) to see that $k(G) \leqslant f(n)\,p^n$, as required.

It is now clear that we proved the result mentioned in the introduction.

THEOREM 1.5. *There exists an integer-valued function $f$ with the property that if $B$ is a $p$-block of defect $d$ of any $p$-solvable group, then $k(B) \leqslant f(d)\,p^d$.*

The function $f$ can be improved if we are dealing with the class of solvable groups.

THEOREM 1.6. *Let $G$ be a solvable group and let $B$ be a $p$-block of $G$ of defect $d$. Then $B$ contains at most $p^d 2^{4d/3} 3^{10d/9}$ irreducible characters.*

*Proof.* According to a theorem of Dornhoff, [5, 36.4], if $H$ is an $n$-dimensional solvable complex linear group, $H$ has a normal abelian subgroup of index not exceeding $g(n) = 2^{4n/3}3^{10n/9}$. The theorem follows from 1.4 and 1.3.

This upper bound for $k(B)$ is larger than the Brauer–Feit upper bound (which applies to an arbitrary group) for primes $p \leqslant 5$. In the next section, we will improve the bound for the prime 2 and for groups of odd order.

## 2

The aim of this section is to investigate in greater detail the action of a $p'$-linear group of odd order (whose solvability we will tacitly assume) on a vector space over $GF(p)$. As a consequence, we will sharpen the bound of 1.5. We begin by applying the methods devised by Dixon in [4] to bound the index of the Fitting subgroup of an odd-order linear group of degree $n$ as a function of $n$. We first prove a version of Dixon's Theorem 3.

LEMMA 2.1. *Let $H$ be an odd order subgroup of the symmetric group of degree $n$. Then $|H| \leqslant 3^{(n-1)/2}$.*

*Proof.* We proceed by induction on $n$. Following the proof of Theorem 3 of [4], we can assume that $H$ is a primitive permutation group. Let $A$ be a minimal normal subgroup of $H$, with $|A| = p^k$. Then $A$ is self-centralizing in $H$ and $n = p^k$. It follows that $|H : A| \leqslant |GL(k, p)|$ and so $|H| \leqslant p^{k+k^2}$ (since $|GL(k, p)| < p^{k^2}$).

Now if $p \geqslant 11$, it can be proved by induction on $k$ that $p^{k+k^2} < 3^{(p^k-1)/2}$ for $k \geqslant 1$. If $p = 5$ or 7, the same inequality holds for $k \geqslant 2$, and if $p = 3$, the inequality holds for $k \geqslant 3$. It is easy to check that our theorem is true in the exceptional cases. Thus the result holds in general.

We can proceed to prove a result analogous to Dixon's Theorem 1 for groups of odd order.

LEMMA 2.2. *Let $H$ be a completely reducible subgroup of odd order of the general linear group $GL(n, E)$ over a field $E$. Let* Fit$(H)$ *denote the Fitting subgroup of $H$. Then $|H : \mathrm{Fit}(H)| \leqslant 3^{(n-1)/2}$.*

*Proof.* If we follow the proof of Theorem 1 in [4] and use 2.1 above, we can suppose that $E$ is algebraically closed and that $H$ is a primitive irreducible linear group of degree $n$. Let $Z$ denote the center of $H$ and let $A/Z$ be a maximal normal abelian subgroup of $H/Z$. Evidently $A$ is contained in Fit$(H)$. By a theorem of Suprunenko, [3, Theorem 4.4], $H/A$ is isomorphic to a subgroup of the direct product of $k$ symplectic groups $Sp(2m_i, q_i)$, $i = 1,..., k$, where $q_1 ,..., q_k$ are distinct primes and $q_1^{m_1} \cdots q_k^{m_k}$ divides $n$. Thus if we can show that whenever $q$ is an odd prime, the order of an odd-order subgroup of $Sp(2m, q)$ is bounded by $3^{(q^m-1)/2}$, our theorem will follow.

We have $|Sp(2m, q)| < q^{2m^2+m}$. It can be proved by induction on $m$ that $q^{2m^2+m} < 3^{(q^m-1)/2}$ for all $m \geqslant 1$, provided that $q > 13$. The same inequality holds for $m \geqslant 2$ if $q > 5$; for $m \geqslant 3$ if $q > 3$; for $m \geqslant 4$ if $q = 3$. The only slightly difficult cases to be checked by direct calculation are $q = 3$, $m = 2$ or 3, and $q = 5$, $m = 2$. Here, the fact that we are dealing with a subgroup of odd order must be taken into account to verify the stated bounds. This completes the proof.

A similar proof yields the following result (which is dependent on working over the field $GF(2)$).

LEMMA 2.3. *Let $H$ be a subgroup of odd order in $GL(n, 2)$. Then $H$ has an abelian normal subgroup of index not exceeding* $3^{(n-1)/2}$.

An immediate consequence of this lemma is an improvement of 1.6 for the prime 2.

THEOREM 2.4. *Let $G$ be a solvable group and let $B$ be a 2-block of $G$ of defect $d$. Then* $k(B) \leqslant 2^d 3^{(d-1)/2}$.

We proceed to obtain a version of 2.4 valid for groups of odd order. We will work with nilpotent subgroups, rather than abelian subgroups.

LEMMA 2.5. *Let $H$ be a nilpotent group of odd order and let $V$ be a faithful irreducible $EH$-module, where $E$ is a field of characteristic coprime to $|H|$. Then, unless $H$ is cyclic, $V$ is an imprimitive $EH$-module.*

*Proof.* We can assume that $H$ is not cyclic. It follows that $H$ has a normal elementary abelian subgroup of order $p^2$ for some prime divisor $p$ of $|H|$ [11, 7.5, p. 303]. Now it is a consequence of Clifford's theorem and Schur's lemma that if $V$ is a primitive $EH$-module, any normal abelian subgroup of $H$ is cyclic. Since we know that this is not the case, $V$ is imprimitive, as required.

LEMMA 2.6. *Let $H$ be a nilpotent group of odd order and let $V$ be a faithful $EH$-module, where $E$ is a field of odd characteristic, coprime to $|H|$. Then there is a vector $v$ in $V$ with $hv \neq v$ for all nonidentity elements $h$ of $H$.*

*Proof.* We proceed by induction on $|H|$. Let us suppose first of all that $V$ is irreducible. If $V$ is a primitive $EH$-module, $H$ is cyclic by 2.5 and the result follows from previous arguments. Thus we can assume that $V$ is imprimitive, so that there is a normal subgroup $M$ of $H$ of prime index $p$ and an irreducible $EM$-module $U$ with $U^H = V$. By induction, there is a vector $u$ in $U$ with $mu \neq u$

for all elements $m$ not in the kernel of the representation of $M$ on $U$. Let $x_1, \ldots, x_p$ be a set of coset representatives of $M$ in $H$. Let

$$v = x_1 \otimes u + \cdots + x_{p-1} \otimes u - x_p \otimes u$$

be a vector in $U^H = V$. As $H$ has odd order and $E$ has odd characteristic, there is no element $m$ of $M$ with $mu = -u$. Given this, it can be checked that $hv \neq v$ for any nonidentity element $h$ of $H$. This proves the lemma when $V$ is irreducible. Complete reducibility of $V$ as an $EH$-module can be invoked to finish the proof in the case that $V$ is reducible.

If we combine 2.2 and 2.6 with the methods of the first section, we see that we have obtained the following result on the number of characters in a $p$-block of a group of odd order.

THEOREM 2.7. *Let $G$ be a group of odd order and let $B$ be a $p$-block of $G$ of defect $d$. Then $k(B) \leqslant p^d 3^{(d-1)/2}$. If $G$ has a nilpotent Sylow $p$-complement, $k(B) \leqslant p^d$.*

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